NUMBER OF VERTICES OF DEGREE THREE IN SPANNING 3-TREES IN SQUARE GRAPHS*

Win Min Aye[1], Liming Xiong[2], Tao Tian[3]

Abstract

In this paper, we show that the square graph of a tree T has a spanning tree of maximum degree at most three and with at most $\max\left\{0, \sum_{x\in W_3(T)} (t_T(x)-2)-2\right\}$ vertices of degree three, where $W_3(T)=\{x\in V(T):$ there are at least three edge-disjoint paths of length at least two that start $x\}$ and $t_T(x)$ is the number of edge-disjoint paths with length at least two that start at a vertex *x*.

Keywords: Square graph; 3-tree; spanning tree.

Introduction

For graph-theoretic notation not explained in this paper, we refer the reader to J. A. Bondy and U. S. R. Murty, 2008. We consider only simple graph in this paper. Let $G=(V,E)$ be a graph with vertex set V and edge set E. A k-tree is a tree with the maximum degree at most k. A graph is called *hamiltonian* (*traceable*, respectively) if it has a *spanning cycle* (*path*, respectively). Thus a graph is traceable if and only if it has a spanning 2-tree. Therefore, the minimum number of vertices of degree three in a spanning 3-tree F of a graph G shows how closed to be traceable the graph G is.

The classic condition for a graph to be traceable is the minimum degree condition, see O. Ore, 1960. It has been extended to consider whether a graph has a spanning k-tree, see S. Win, 1979, in references. It has also been extended to the condition for the existence of a spanning tree with at most k leaves, see H. J. Broersma and H. Tuinstra, 1998. H. J. Broersma and H. Tuinstra gave more structures of the graphs satisfying the condition given by S. Win, 1979; M. Aung and A. Kyaw, 1998, considered the maximum k-tree. V. Neumann-Lara and E. Rivera-Compo, 1991, gave an independence number condition for a graph to have a spanning k-tree with bounded number

1. Dr, Professor, Department of Mathematics, Kalay University.
2. Dr, Professor, School of Mathematics and Statistics, Beijing Institute of Technology.
3. Ph.D Candidate, School of Mathematics and Statistics, Beijing Institute of Technology.
* Best Paper Award Winning Paper in Mathematic, (2018)

of vertices with degree *k*, for $k \geq 4$. M. Tsugaki, 2009, gave a similar condition for $k = 3$.

The *square graph* of a graph *G*, denoted by G^2, is the graph with $V(G^2) = V(G)$ in which two vertices are adjacent in G^2 if their distance in *G* is at most two. Thus $G \subseteq G^2$. H . Fleischner, 1974, proved that the square graph of a 2-connected graph is hamiltonian, which was extended by G. Hendry and W. Vogler, 1985. Y. Caro, I. Krasikov and Y. Roditty, 1991, showed that the square graph of a connected graph has a spanning 3-tree.

Motivated by the results given above and by the observation that the minimum number of vertices of degree three in a spanning 3-tree *F* of a graph G^2 may measure how closed to be traceable the graph G^2 is, Q. Wu, 2016, showed that the square graph of a tree *T* has a spanning 3-tree *F* in which every leaf of *T* has degree one or two and *F* has at most $\max\left\{0, \min\left\{\left\lfloor \frac{n-p(T)+3}{2} \right\rfloor, \left\lfloor \frac{n-5}{2} \right\rfloor\right\}\right\}$ vertices of degree three where *p(F)* is the length of the longest path of *F*. In the whole paper, we let *p(T)* be the length of a longest path of a tree *T*.

Theorem 1

Let *G* be a connected graph of order *n*. Then G^2 has a 3-tree *F* with at most

$$\min_{T \subseteq G} \max\left\{0, \min\left\{\left\lfloor \frac{n-p(T)+3}{2} \right\rfloor, \left\lfloor \frac{n-5}{2} \right\rfloor\right\}\right\}$$

vertices of degree three.

In this paper, we intend to improve the result above. Firstly, we give the following definitions. Let *T* be a tree of order *n* and *x* a vertex of *T*. We define $t_T(x)$ to be the number of edge-disjoint paths with length at least two that start at a vertex *x* and $W_3(T) = \{x \in V(T):$ there are at least three edge-disjoint paths of length at least two starting at $x\}$.

Obviously, $t_T(x) \le d_T(x)$ for any vertex *x* of *T*, where $d_T(x)$ denote the degree of *x* in *T*. For example, for a star $T = K_{1,k}$, it holds that $t_T(x) = 0$ and $d_T(x) = k$ for the center vertex *x* of $K_{1,k}$. From the definition of $W_3(T)$, one may obtain the following observation.

Observation 2

Let *T* be a tree of order *n*. Then

$$\sum_{x \in W_3(T)} t_T(x) - 2|W_3(T)| = \sum_{x \in W_3(T)} (t_T(x) - 2) \le \frac{n - p(T) - 1}{2}.$$

Proof

Let P_0 be a longest path of *T*. Then we may obtain *T* from P_0 by adding a path P_i of *T* such that P_i has a leaf of *T*, iteratively. In order to increase $\sum_{x \in W_3(T)} (t_T(x) - 2)$ at least one, these P_i (note that, in each step of the proceeding of adding path with a leaf of *T*, two leaves distance of at least two may be counted once in $\sum_{x \in W_3(T)} (t_T(x) - 2)$ should have length at least two. Therefore, Observation 2 follows.

By Observation 2, in this paper, we continue to give an upper bound for the number of vertices of degree three of spanning 3-tree *F* in square graph G^2 as

$$\min_{T \subseteq G} \max \left\{ 0, \sum\nolimits_{x \in W_3(T)} t_T(x) - 2|W_3(T)| - 2 \right\}$$

where *T* is a spanning tree of *G*. Hence

Theorem 3

Let *G* be a connected graph of order *n*. Then G^2 has a spanning 3-tree *F* with at most

$$\min_{T \subseteq G} \max \left\{ 0, \sum_{x \in W_3(T)} t_T(x) - 2|W_3(T)| - 2 \right\}$$

vertices of degree three, where *T* is a spanning tree of *G*.

Observation 2 shows that the bound in the theorem above improves the one gave in Theorem 1. In the next section, we shall give some auxiliary results, which will be used to proof of Theorem 3 in Section 4. In the last section, we shall show the sharpness of Theorem 3 and Observation 2 and also compare two upper bounds in Theorems 1 and 3, respectively.

Preliminaries and Auxiliary Results

For $S \subseteq V(G)$ or *E(G)*, we denote by $G[S]$ the subgraph of *G* induced by *S*. For a positive integer *s*, the graph $S(K_{1,s})$ is obtained from the complete bipartite graph $K_{1,s}$ by subdividing each edge once. The graph *G* is said to be $S(K_{1,3})$-free if it does not contain any induced copy of $S(K_{1,3})$. We use $N_G(u)$ and $d_G(u)$ to denote the neighbors and the degree of *u* in *G*. A *leaf* or *pendant vertex* is a vertex of degree one in a graph. A tree *T* is called a *caterpillar* if by deleting all pendant vertices of *T* we get a path. The following results will be used in our proofs.

Theorem 4

If *G* is a connected $S(K_{1,3})$-free graph of order at least three, then G^2 is hamiltonian.

Corollary 5

If *G* is a connected $S(K_{1,3})$-free graph, then G^2 has a hamiltonian path starting at any vertex of *G*.

Let $n_1(T), n_2(T)$ and $n_3(T)$ denote the number of vertices of degrees 1, 2 and 3 in a 3-tree *T* of order *n*, respectively. We have

$$n_1(T) + n_2(T) + n_3(T) = n \qquad (1)$$

and

$$n_1(T) + 2n_2(T) + 3n_3(T) = 2|E(T)| = 2(n-1), \qquad (2)$$

one may obtain that $n_2(T) + 2n_3(T) = n - 2,$ and that

$$n_1(T) = n_3(T) + 2. \qquad (3)$$

A well known and easily proved equality for 3-trees: they are the same as $n_2(T) = 0$ in the above equations, one can then show that $n_1(T) = n_3(T) + 2 = \frac{n - n_2(T) + 2}{2}$. So we only need to consider the upper bound of $n_3(T)$ for a 3-tree *T*.

Lemma 6

If $|W_3(T)| = 0,$ then T^2 has a hamiltonian path starting at any vertex.

Proof

Since $|W_3(T)| = 0, T$ is $S(K_{1,3})$-free. Then by Corollary 5, the lemma holds.

Lemma 7

For each caterpillar *T* (*i.e.*, $|W_3(T)| = 0$), T^2 has a spanning path starting at one end vertex *u* of longest path of *T* and ending at neighbor of *u*.

Proof

We prove by induction on $|V(T)|$. It is obvious when $|V(T)| \leq 4.$ Suppose that it holds when $|V(T)| \leq n-1 (n \geq 5).$ We now consider the case when $|V(T)| = n.$ We choose a longest path *P* of *T* and take an end vertex *u* of *P* and let $v \in N_p(u).$ Let $T_1 = T - u.$ Then $|V(T_1)| = n-1.$ By induction hypothesis and when $d_T(v) = 2, T_1^2$ has a spanning path Q starting at vertex *v* and ending at *h*, where $h \in N_T(v) \setminus \{u\}.$ Thus $Q \cup \{uh\}$ is a spanning path starting at vertex *u* and ending at vertex *v* in T^2. By induction hypothesis and when $d_T(v) \geq 3, T_1^2$ has a spanning path Q_1 starting at vertex *w* and ending at vertex *v*, where $w \in N_T(v) \setminus \{u\}$ with $d_T(w) = 1.$ Thus $Q_1 \cup \{uw\}$ is a spanning path starting at vertex *u* and ending at vertex *v* in T^2.

Lemma 8

Let T be a tree of order n with $W_3(T)=\{u\}$. Then T^2 has a spanning 3-tree F with $n_3(F)=t_T(u)-2$ such that $d_F(u)=1$ and each leaf in T has degree at most two in F.

Proof

Let the neighbors of u be labeled by $u_1,\ldots,u_t,u_{t+1},u_{t+2},\ldots,u_{d_T}(u)$ such that $d_T(u_i)\geq 2$ for $1\leq i\leq t$ and $d_T(u_i)=1$ for $t+1\leq i\leq d_T(u)$. Let T_i be the component of $T-u$ with at least two vertices and $u_i\in V(T_i)(1\leq i\leq t)$. Since $W_3(T)=\{u\}$, $W_3(T_i)=$ Ø. Let $T_u=T_1\cup\{uu_1\}$. Then by Lemma 7, T_u^2 has spanning path Q_u starting at vertex u and ending at vertex u_1. By Lemma 6, T_i^2 $(2\leq i\leq t)$ has a spanning path Q_i starting at u_i .

Then $F=T^2\left[E(\mathrm{Q_u})\cup(\cup_{i=2}^{t}E(Q_i))\cup E(u_1u_{t+1}u_{t+2}\ldots u_{d_T(u)}u_2u_3\ldots u_t)\right]$ is a spanning 3-tree in T^2. This implies that $n_3(F)=t_T(u)-2$ such that $d_F(u)=1$ and each leaf in T has degree at most two in F.

Lemma 9

The following statements hold:

(i) If $W_3(T)=\{u\}$, then T^2 has a spanning 3-tree F with $n_3(F)\leq\max\{0,t_T(u)-4\}$ such that each leaf in T has degree at most two in F.

(ii) If $|W_3(T)|=2$, then T^2 has a spanning 3-tree F with $n_3(F)=\sum_{x\in W_3(T)}t_T(x)-6$ such that each leaf in T has degree at most two in F.

Proof

Suppose that $W_3(T)=\{u\}$. Then by Lemma 6, it is easy to show that T^2 with $t_T(u)\leq 4$ is traceable. In the following, we assume that $t_T(u)\geq 5$. Let the neighbors of u be labeled by $u_1,\ldots,u_t,u_{t+1},u_{t+2},\ldots,u_{d_T(u)}$ such that $d_T(u_i)\geq 2$ for $1\leq i\leq t$ and $d_T(u_i)=1$ for $t+1\leq i\leq d_T(u)$. Let T_i be the component of $T-u$ with at least two vertices and $u_i\in V(T_i)$ $(1\leq i\leq t)$. Then $W_3(T_i)=\varnothing$. Let $T_{12}=T_1\cup T_2\cup\{uu_1,uu_2\}$ and $T_{34}=T_3\cup T_4\cup\{uu_3,uu_4\}$.

Obviously, T_{12} and T_{34} are both caterpillars. Then by Lemma 6, T_{12}^2 and T_{34}^2 have a spanning path P_l and P_s starting at vertex u, respectively. By Lemma 6, T_i^2 $(5\leq i\leq t)$ has a spanning path Q_i starting at vertex u_i. Then $F=T^2\left[E(P_l)\cup E(P_s)\cup\left(\cup_{i=5}^{t}E(Q_i)\right)\cup E(uu_{t+1}u_{t+2}\ldots u_{d_T(u)}u_5u_6\ldots u_t)\right]$ is a spanning 3-tree in T^2. This implies that $n_3(F)=t_T(u)-4$ such that each leaf in T has degree at most two in F.

Suppose that $W_3(T)=\{u,v\}$ and $uv\in E(T)$. Let T_1 and T_2 be two components of $T-\{uv\}$. Then by Lemmas 6 and 8, T_1^2 has a spanning 3-tree F_1 with $n_3(F_1)=t_T(u)-3$ and $d_{F_1}(u)=1$, T_2^2 has a spanning 3-tree F_2 with $n_3(F_2)=t_T(v)-3$ and $d_{F_2}(v)=1$. Then $F=F_1\cup F_2\cup\{uv\}$ is a spanning 3-tree of T with $n_3(F)=\sum_{x\in W_3(T)}t_T(x)-6$ and each leaf in T has degree at most two in F.

Suppose that $W_3(T)=\{u,v\}$ and $uv\notin E(T)$. Obviously, u and v are connected by path P. We assume that $uw,vw'\in E(P)$ (may $w=w'$). Let T_1 and T_2 be the component of $T-\{uw,vw'\}$ containing vertex u and v, respectively. Then by Lemmas 6 and 8, T_1^2 has a spanning 3-tree F_1 with $n_3(F_1)=t_T(u)-3$ and $d_{F_1}(u)=1$, T_2^2 has a spanning 3-tree F_2 with $n_3(F_2)=t_T(v)-3$ and $d_{F_2}(v)=1$. Let $T_0=(T-T_1-T_2)\cup\{uw,vw'\}$. Since T_0 is a caterpillar, T_0^2 has a spanning path Q with end vertices u and v. Then $F=F_1\cup F_2\cup Q$ is a 3-tree of T^2 with $n_3(F)=\sum_{x\in W_3(T)}t_T(x)-6$ and each leaf in T has degree at most two in F

Proof of Theorem 3

In this section, we present the proof of Theorem 3. In order to prove Theorem 3, we only need to show the following result.

Theorem 10

Let T be a tree. Then T^2 has a spanning 3-tree F with at most

$$\max\left\{0, \sum_{x\in W_3(T)} t_T(x) - 2|W_3(T)| - 2\right\}$$

vertices of degree three such that each leaf in a spanning tree T has degree at most two in F.

Now, we may present the proof of Theorem 10.

Proof of Theorem 10

We prove this theorem by induction on $|W_3(T)|$. If $|W_3(T)| \le 2$, then by Lemmas 7 and 9, the theorem holds. Suppose that the theorem holds when $|W_3(T)| < k$ $(k \ge 3)$.

In the following, we only need to show that the conclusion of Theorem 10 holds for the case when $|W_3(T)| = k$.

We may choose one pair of vertices $\{u,v\}$ where $u \in W_3(T)$ and $v \in N_T(u)$ such that $|W_3(T_1)| \le 1$ and $|W_3(T_2)| \le |W_3(T)| - 1$, where T_1 is a component of $T - \{uv\}$ and $T_2 = (T - T_1) \cup \{uv\}$.

By Lemmas 6 and 8, T_1^2 has a spanning 3-tree F_1 such that $n_3(F_1) = t_{T_1}(u) - 2 = t_T(u) - 3$ and $d_{F_1}(u) = 1$. Let F_2 be a spanning 3-tree of T_2^2.

By induction, $n_3(F_2) \le \max\left\{0, \sum_{x\in W_3(T_2)} t_{T_2}(x) - 2|W_3(T_2)| - 2\right\}$, and each leaf in T_1 and T_2 has degree at most two in F_1 and F_2, respectively. Then by $d_{T_2}(u) = 1$, $d_{F_2}(u) \le 2$. Let $F = F_1 \cup F_2$. Since $d_{F_1}(u) = 1$, F is a spanning 3-tree of T^2 with $n_3(F) \le n_3(F_1) + n_3(F_2) + 1$ such that each leaf in T has degree at most two in F.

We distinguish the following three cases to obtained our results.

Case 1

$t_T(v)=2.$

Then $W_3(T)=W_3(T_2)\cup\{u\}.$ Note that $\sum_{x\in W_3(T_2)} t_{T_2}(x)-2|W_3(T_2)|-2\geq 0.$ Therefore,

$$n_3(F)\leq n_3(F_1)+n_3(F_2)+1$$

$$\leq t_T(u)-3+\max\left\{0, \sum_{x\in W_3(T_2)} t_{T_2}(x)-2|W_3(T_2)|-2\right\}+1$$

$$=t_T(u)-3+\left(\sum_{x\in W_3(T)\setminus\{u\}} t_T(x)-2\left(|W_3(T)|-1\right)-2\right)+1$$

$$=\sum_{x\in W_3(T)} t_T(x)-2|W_3(T)|-2.$$

Case 2

$t_T(v)=3.$

Then $W_3(T)=W_3(T_2)\cup\{u,v\}.$

Note that possible $\sum_{x\in W_3(T_2)} t_{T_2}(x)-2|W_3(T_2)|-2=-1,$ however, $\sum_{x\in W_3(T_2)} t_{T_2}(x)-2|W_3(T_2)|-2+1\geq 0.$ Therefore,

$$n_3(F)\leq n_3(F_1)+n_3(F_2)+1$$

$$n_3(F)\leq t_T(u)-3+\max\left\{0, \sum_{x\in W_3(T_2)} t_{T_2}(x)-2|W_3(T_2)|-2\right\}+1$$

$$\leq t_T(u)-3+\max\left\{0, \sum_{x\in W_3(T_2)} t_{T_2}(x)-2|W_3(T_2)|-2+1\right\}+1$$

$$=t_T(u)+(t_T(v)-3)-3$$

$$+\left(\sum_{x\in W_3(T)\backslash\{u,v\}} t_T(x)-2\left(\left|W_3(T)\right|-2\right)-1\right)+1$$
$$=\sum_{x\in W_3(T)} t_T(x)-2\left|W_3(T)\right|-2.$$

Case 3

$t_T(v)\geq 4.$

Then $W_3(T)=W_3(T_2)\cup\{u\}$. Note that $t_{T_2}(v)=t_T(v)-1$ and $\sum_{x\in W_3(T)\backslash\{u\}} t_T(x)-1-2(\left|W_3(T)\right|-1)-2\geq 0$. Therefore,

$$n_3(F)\leq n_3(F_1)+n_3(F_2)+1$$
$$\leq t_T(u)-3+\max\left\{0,\sum_{x\in W_3(T_2)} t_{T_2}(x)-2\left|W_3(T_2)\right|-2\right\}+1$$
$$=t_T(u)-3+\max\left\{0,\sum_{x\in W_3(T)\backslash\{u\}} t_T(x)-1\right.$$
$$\left.-2(\left|W_3(T)\right|-1)-2\right\}+1$$
$$=t_T(u)-3+\left(\sum_{x\in W_3(T)\backslash\{u\}} t_T(x)-2\left|W_3(T)\right|-1\right)+1$$
$$=\sum_{x\in W_3(T)} t_T(x)-2\left|W_3(T)\right|-3$$
$$<\sum_{x\in W_3(T)} t_T(x)-2\left|W_3(T)\right|-2.$$

In all cases, $n_3(F)\leq \max\left\{0,\sum_{x\in W_3(T)} t_T(x)-2\left|W_3(T)\right|-2\right\}$. This proves the theorem for the case when $\left|W_3(T)\right|=k$. Therefore, by induction, the theorem holds.

Concluding Remarks

Firstly, we have the following remark that shows the sharpness of the bound in Theorem 3.

Remark 11

The upper bound in Theorem 3 is sharp, because the square of the tree T with $W_3(T)=\{u\}$ and $t_T(u)=5$ has no hamiltonian path.

Observation 2 shows that the upper bound in Theorem 3 is better than Theorem 1. On the other hand, we may construct many examples to show that those two bounds in Theorems 1 and 3 may have many different. To see this, we let T_0 be a tree that is composed of k pathes of length exactly $l \geq 2$ (we may take $l = k$) with a common vertex and the length of a longest path of T_0 is $2l$ (i.e., the tree obtained by subdividing all edges in the star $K_{1,k}$ exactly $l-1$ times).

Then

$$|V(T_0)| = lk + 1$$

$$\sum_{x \in W_3(T_0)} (t_{T_0}(x) - 2) - 2 = k - 4$$

and

$$\frac{|V(T_0)| - p(T_0) + 3}{2} = \frac{lk + 1 - 2l + 3}{2} = \frac{lk - 2l + 4}{2}.$$

From the equations above, one may know that the different $\frac{lk-2l+4}{2} - (k-4) = \frac{lk-2l-2k+12}{2}\left(= \frac{(k-2)^2+8}{2} \text{ if } l = k \geq 3, \text{ respectively}\right)$ between the two upper bounds in Theorems 1 and 3, respectively, may be any large $\left(\frac{(k-2)^2+8}{2} \text{ if } l = k \geq 5, \text{ respectively}\right)$.

Finally, we show that the inequality in Observation 2 is also sharp. To see this, we construct a tree as follows: we use T_0 to denote the resulting graph obtained a path P_0 by attaching at least one pendant edge on each vertex of P_0. Now we obtain the graph T_0' from T_0 by subdividing these pendant edges exactly once. Then $\sum_{x\in W_3(T_0')}(t_{T_0'}(x)-2)=\frac{|V(T_0')|-p(T_0')-1}{2}$ (here we suppose that $|V(T_0')|-p(T_0')-1$ is even). The sharpness shows that Observation 2 is itself interesting.

Acknowledgements

The work is supported by the Natural Science Funds of China (No: 11471037, 11671037).

References

G., Hendry and W., Vogler (1985), The Square of a Connected $S(K_{1,3})$-Free Graph is Vertex Pancyclic, *J. Graph Theory*, 9, p.535-537.

H., Fleischner (1974), The Square of Every Two-Connected Graph is Hamiltonian, *J. Combinatorical Theory Ser*. B, 16, p. 29-34.

H.J., Broersma and H., Tuinstra (1998), Independence Trees and Hamiltonian Cycles, *J. Graph Theory*, 29, p. 227-237.

J. A., Bondy, and U. S. R., Murty (2008), *Graph Theory*, Graduate in Mathematics, Vol. 244, Springer.

M., Aung and A., Kyaw (1998), Maximal Trees with Bounded Maximum Degree in a Graph, *Graphs Combin*., 14, p.209-221.

M., Tsugaki (2009), A Note on a Spanning 3-Tree, *Combinatorica* 29(1), p.127-129.

O., Ore (1960), Note on Hamiltonian Circuits, *Amer. Math.Monthly*, 67, p.55.

Q., Wu, (2016), Spanning Trees with Minimum Number of Leaves in the Square Graph of a Tree, *Journal of Computational Methods in Sciences and Engineering*, 16(1), p.21-27.

S., Win (1979), On a Conjecture of Las Vergnas Concerning Certain Spanning Trees in Graphs, *Resultate Math*.2, p.215-224.

V., Neumann-Lara and E., Rivera-Compo (1991), Spanning Trees with Bounded Degrees, *Combinatorica*, 11(1), p. 55-61.

Y., Caro, I., Krasikov and Y., Roditty (1991), On the Largest Tree of Given Maximum Degree in a Connected Graph, *J. Graph Theory*, 15, p. 7-13.